

Discrete Anomalies of Binary Groups

Christoph Luhn*

*Institute for Fundamental Theory, Department of Physics,
University of Florida, Gainesville, FL 32611, USA*

Abstract

We derive the discrete anomaly conditions for the binary tetrahedral group \mathcal{T}' as well as the binary dihedral groups \mathcal{Q}_{2n} . The ambiguities of embedding these finite groups into $SU(2)$ and $SU(3)$ lead to various possible definitions of the discrete indices which enter the anomaly equations. We scrutinize the different choices and show that it is sufficient to consider one particular assignment for the discrete indices. Thus it is straightforward to determine whether or not a given model of flavor is discrete anomaly free.

*E-mail: luhn@phys.ufl.edu

1 Introduction

The family structure of the Yukawa couplings which give rise to the masses and mixings of quarks and leptons remains unexplained within the Standard Model (SM). One of the most successful ideas to overcome this annoying shortcoming consists in extending the SM gauge group with a family-dependent $U(1)$ symmetry. These Froggatt-Nielsen [1] models seem to be well suited for addressing the hierarchies of the quark and charged lepton masses as well as the small angles of the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix. However, they fail to give a convincing account of the neutrino sector which features either no or only a very mild hierarchy.

The observation that the Maki-Nakagawa-Sakata-Pontecorvo (MNSP) mixing matrix, to a very good approximation, exhibits the so-called tri-bimaximal structure [2] has added its share to the mystery surrounding the fermionic masses and mixings. Spurred on by this remarkable fact, model builders have resorted to imposing an underlying non-Abelian finite family symmetry. As there are three families of quarks and leptons, the finite group should have two- or three-dimensional irreps. This requirement limits the candidates to the finite subgroups of $SU(3)$, $SO(3)$, and $SU(2)$.¹ Having picked a preferred finite group \mathcal{G} , one is still left with the choice of assigning the various particles of a model to irreps of \mathcal{G} .

In order to constrain the possibilities, Ref. [3] exploits the conditions arising from requiring that \mathcal{G} should originate in an anomaly free gauge symmetry, a framework which stabilizes the discrete symmetry against violation by quantum gravity effects [4]. The formulation of the discrete anomaly conditions necessitates the definition of Dynkin-type indices for the irreps of finite groups. These so-called discrete indices have to be defined individually for each group \mathcal{G} . Focusing on several popular finite family groups, the general procedure of obtaining the discrete anomaly conditions has been established in Ref. [3]. In all cases, \mathcal{G} was a subgroup of $SU(3)$ or $SO(3)$ but not $SU(2)$. Furthermore, the embedding into the continuous group was always defined uniquely.²

It is the purpose of this letter to discuss the constraint arising from the discrete anomaly $\mathcal{G} - \mathcal{G} - U(1)_Y$ for some finite subgroups of $SU(2)$ which have been put forward as possible family symmetries, notably the binary tetrahedral group \mathcal{T}' [5–10] and the binary dihedral groups \mathcal{Q}_{2n} [11–21]. The fact that subgroups of $SU(2)$ necessarily have two-dimensional irreps appears to have some advantage since a $2 + 1$ structure can naturally separate the third family of fermions from the other two. In the process of deriving the discrete anomaly conditions we will find that there exist different embeddings which seem to make a consistent definition of the discrete indices impossible. It is one of the main intentions of this letter to shed some light on this ambiguity, proving that *one particular assignment of discrete indices is sufficient* to determine whether

¹An embedding of these finite groups into bigger continuous groups is possible in principle, but it would not be a genuine one.

²In the case of \mathcal{D}_5 there are actually two distinct embeddings. However, they are equivalent as they can be related to each other by relabeling the two-dimensional irreps $\mathbf{2}_1 \leftrightarrow \mathbf{2}_2$.

a specific model of flavor is discrete anomaly free or not. We illustrate the procedure by applying our results to some existing examples.

2 The Binary Tetrahedral Group \mathcal{T}'

The alternating group on four letters, \mathcal{A}_4 , is the symmetry group of the tetrahedron, and as such a subgroup of $SO(3)$. It has three one-dimensional and one three-dimensional irreps. Similar to $SU(2)$ being the double cover of $SO(3)$, the binary tetrahedral group \mathcal{T}' is the double cover of \mathcal{A}_4 . It is a subgroup of $SU(2)$ and has three two-dimensional irreps in addition to those of \mathcal{A}_4 . A convenient way to define this group is provided in terms of its presentation [7, 22]

$$\langle r, s, t \mid r^2 = t^3 = (st)^3 = 1, s^2 = r, rt = tr \rangle . \quad (1)$$

Here we have introduced the auxiliary generator r in order to manifest the connection between the binary tetrahedral group and the alternating group: setting $r = 1$ in Eq. (1) yields the presentation of \mathcal{A}_4 . The one- and three-dimensional irreps of \mathcal{T}' are identical to the irreps of \mathcal{A}_4

$$\begin{aligned} \mathbf{1}_k : \quad s &= 1, \quad t = \omega^k, \\ \mathbf{3} : \quad s &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad t = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \end{aligned}$$

while the two-dimensional irreps take the form

$$\mathbf{2}_k : \quad s = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad t = \frac{\omega^k}{\sqrt{2}} \begin{pmatrix} \tau^3 & \tau \\ \tau^3 & \tau^5 \end{pmatrix},$$

with $\omega = e^{2\pi i/3}$, $\tau = e^{\pi i/4}$ and $k = 0, 1, 2$. The resulting Kronecker products are those of \mathcal{A}_4

$$\mathbf{1}_k \otimes \mathbf{1}_l = \mathbf{1}_{k+l}, \quad \mathbf{1}_k \otimes \mathbf{3} = \mathbf{3}, \quad \mathbf{3} \otimes \mathbf{3} = \mathbf{1}_0 + \mathbf{1}_1 + \mathbf{1}_2 + 2 \cdot \mathbf{3},$$

plus the ones involving the doublets

$$\mathbf{2}_k \otimes \mathbf{1}_l = \mathbf{2}_{k+l}, \quad \mathbf{2}_k \otimes \mathbf{2}_l = \mathbf{1}_{k+l} + \mathbf{3}, \quad \mathbf{2}_k \otimes \mathbf{3} = \mathbf{2}_0 + \mathbf{2}_1 + \mathbf{2}_2.$$

The subscripts are understood modulo 3, so that the complex conjugates of $\mathbf{1}_k$ and $\mathbf{2}_k$ are given as $\overline{\mathbf{1}}_k = \mathbf{1}_{-k}$ and $\overline{\mathbf{2}}_k = \mathbf{2}_{-k}$. Writing the spinor of the irrep $\mathbf{2}_k$ as $\begin{pmatrix} u \\ v \end{pmatrix}$, the conjugated spinor $\begin{pmatrix} v^* \\ -u^* \end{pmatrix} = i\sigma_2 \begin{pmatrix} u^* \\ v^* \end{pmatrix}$ transforms with the same matrices as the irrep $\mathbf{2}_{-k}$. Notice that for $k = 0$ this is similar to the $\mathbf{2}$ of $SU(2)$ being its own conjugate.

Since the determinant³ of t is ω^{2k} for the two-dimensional irreps, there is only one embedding of \mathcal{T}' into $SU(2)$: the $\mathbf{2}$ of $SU(2)$ is identified with the $\mathbf{2}_0$ of \mathcal{T}' .

³Bear in mind that the generators r, s, t of the finite group shall be elements of the $SU(2)$ Lie group and not of the $SU(2)$ Lie algebra.

Then the decomposition of all other irreps of $SU(2)$ is fixed by the Kronecker products. The spinors break up into sums of $\mathbf{2}_k$, while the vectors decompose into sums of $\mathbf{1}_k$ and $\mathbf{3}$. Hence the two-dimensional irreps of \mathcal{T}' are spinor-like. The following table lists the decomposition of the smallest irreps $\boldsymbol{\rho}$ of $SU(2)$ into irreps \mathbf{r} of \mathcal{T}' . For the purpose of assigning discrete indices to \mathbf{r} , we also display the Dynkin indices $\ell(\boldsymbol{\rho})$.

Irreps $\boldsymbol{\rho}$ of $SU(2)$	Decomposition of $\boldsymbol{\rho}$ under \mathcal{T}'	Dynkin index $\ell(\boldsymbol{\rho})$
2	$\mathbf{2}_0$	1
3	$\mathbf{3}$	4
4	$\mathbf{2}_1 + \mathbf{2}_2$	10
5	$\mathbf{1}_1 + \mathbf{1}_2 + \mathbf{3}$	20
6	$\mathbf{2}_0 + \mathbf{2}_1 + \mathbf{2}_2$	35
7	$\mathbf{1}_0 + 2 \cdot \mathbf{3}$	56

It is easy to show that $\mathbf{1}_1$ and $\mathbf{1}_2$ as well as $\mathbf{2}_1$ and $\mathbf{2}_2$ always come in pairs, so that the discrete indices $\tilde{\ell}(\mathbf{r})$ cannot be defined uniquely. We obtain

$$\begin{aligned}
\tilde{\ell}(\mathbf{1}_0) &= 0, & \tilde{\ell}(\mathbf{1}_1) &= x, & \tilde{\ell}(\mathbf{1}_2) &= 16 - x, \\
\tilde{\ell}(\mathbf{2}_0) &= 1, & \tilde{\ell}(\mathbf{2}_1) &= y, & \tilde{\ell}(\mathbf{2}_2) &= 10 - y, & \tilde{\ell}(\mathbf{3}) &= 4,
\end{aligned} \tag{2}$$

where the parameters x and y can take arbitrary values. Comparing the decomposition of the **6** with those of the **2** and **4**, we see that the discrete indices $\tilde{\ell}(\mathbf{r})$ can only be defined modulo $N_\ell = 24$. Using the methods presented in Ref. [3] one can prove that the assignments in Eq. (2) are consistent for all irreps of $SU(2)$. That is, given an arbitrary $SU(2)$ irrep $\boldsymbol{\rho}$ and its decomposition into irreps of the finite subgroup, the sum of the corresponding *discrete* indices adds up to the Dynkin index of $\boldsymbol{\rho}$ modulo N_ℓ , see also Eq. (2.6) of Ref. [3].

At this point, due to the modulo N_ℓ , it is already evident that the discrete anomaly conditions derived in the following are *necessary but not sufficient* for a given model of flavor to originate from an anomaly free gauged flavor symmetry. In order to guarantee that the continuous high-energy flavor theory is anomaly free, the full theory together with its various breaking mechanisms and the resulting heavy degrees of freedom need to be investigated in detail. Such an endeavor goes well beyond the scope of this letter.

It is worth mentioning that there is another condition (independent from the anomaly discussion) which is necessary in order to gauge the discrete symmetry: The continuous high-energy theory must always, by definition, start with *complete* irreps of $SU(2)$. This, however, does not entail that the *light* irreps of \mathcal{G} have to add up to complete $SU(2)$ irreps, because parts of an original $SU(2)$ irrep might acquire a mass while the rest remains massless. For instance, consider the **5** of $SU(2)$ which decomposes into $\mathbf{1}_1 + \mathbf{1}_2 + \mathbf{3}$ of \mathcal{T}' . The one-dimensional irreps might remain light, while the triplet acquires a mass. This could be achieved by introducing a new **3** of $SU(2)$ which, after the breakdown

to \mathcal{T}' , transforms as a $\mathbf{3}$ of \mathcal{T}' and can then form a bilinear mass term with the triplet of the original $\mathbf{5}$. This situation is in some sense analogous to the doublet-triplet splitting in $SU(5)$ grand unified theories.

Nonetheless, there do exist cases in which it is possible to show that the assignment of the light particle content under the discrete symmetry *cannot* originate from complete multiplets of $SU(2)$ unless additional light degrees of freedom are introduced. This is owed to the breaking pattern of $SU(2)$ down to \mathcal{T}' : the irreps $\mathbf{1}_1$ and $\mathbf{1}_2$ *always* come in pairs, as do the irreps $\mathbf{2}_1$ and $\mathbf{2}_2$. Since mass terms need to be of the form $\mathbf{1}_1 \otimes \mathbf{1}_2$ or $\mathbf{2}_1 \otimes \mathbf{2}_2$, it is impossible to make one constituent of such a pair of irreps heavy while the other remains light (without having a new light field which again would complete the pair). Therefore, the models of Refs. [7–10] are incomplete within an $SU(2)$ framework⁴ already from this perspective.

Turning back to the discussion of the anomalies, one can alternatively embed the binary tetrahedral group into $SU(3)$ instead of $SU(2)$. Identifying the $\mathbf{3}$ of $SU(3)$ with the $\mathbf{3}$ of \mathcal{T}' would only generate the representations of \mathcal{A}_4 , excluding the two-dimensional spinor-like irreps of \mathcal{T}' . We are therefore left with three conceivable embeddings which are defined by:

$$(i) : \mathbf{3} \rightarrow \mathbf{1}_0 + \mathbf{2}_0, \quad (ii) : \mathbf{3} \rightarrow \mathbf{1}_1 + \mathbf{2}_1, \quad (iii) : \mathbf{3} \rightarrow \mathbf{1}_2 + \mathbf{2}_2.$$

As in the case of $SU(2)$, the derived discrete indices are not uniquely determined. For (i), the assignments turn out to be identical to those of Eq. (2). On the other hand, (ii) and (iii) lead to the discrete indices of Eq. (2) with y replaced by $1 - x$.

The question arises: Which indices should be used for the discrete anomaly conditions? Since the low-energy models that adopt a discrete family symmetry do not specify a particular embedding, the most general approach consists in choosing a parameterization of the discrete indices which holds for *any* of the above embeddings. That is, we have to set $y = 1 - x$ in Eq. (2). Consequently, the discrete indices solely depend on the parameter x .

The requirement of massive degrees of freedom not affecting the discrete anomaly conditions further constrains x . Among the particles that have a \mathcal{T}' invariant mass term, only the pairs $\mathbf{1}_1$ and $\mathbf{1}_2$ as well as $\mathbf{2}_1$ and $\mathbf{2}_2$ might give non-zero contributions to the $\mathcal{T}' - \mathcal{T}' - U(1)_Y$ anomaly. With the hypercharges being opposite to each other, the particles of each pair add

$$\begin{aligned} \mathbf{1}_1 - \mathbf{1}_2 : & \quad Y_{\mathbf{1}_1}(2x - 16), \\ \mathbf{2}_1 - \mathbf{2}_2 : & \quad Y_{\mathbf{2}_1}(2y - 10) = -Y_{\mathbf{2}_1}(2x + 8), \end{aligned}$$

to the discrete anomaly. Choosing either $x = 8$ or $x = 20$ their contributions vanish modulo 24 provided that the hypercharges Y_{\dots} are normalized to be

⁴In the case where \mathcal{T}' originates from $SU(3)$, the situation is more involved due to different allowed breaking patterns of the $SU(3)$ irreps. As it is the intention of this letter to discuss the discrete anomaly conditions, we just state that all models of Table 2 are incomplete because of the requirement to start out with complete multiplets of either $SU(2)$ or $SU(3)$. Yet, we show the examples to illustrate the procedure of applying our discrete anomaly conditions.

\mathcal{T}' irreps	$\tilde{\ell}(\mathbf{r})$ [$N_\ell = 24$]
$\mathbf{1}_0$	0
$\mathbf{1}_1$	$8 + \xi \cdot 12$
$\mathbf{1}_2$	$8 + \xi \cdot 12$
$\mathbf{2}_0$	1
$\mathbf{2}_1$	$17 - \xi \cdot 12$
$\mathbf{2}_2$	$17 - \xi \cdot 12$
$\mathbf{3}$	4

Table 1: The discrete indices $\tilde{\ell}$ for the irreps \mathbf{r} of \mathcal{T}' , with $\xi = 0, 1$.

integer.⁵ Therefore a model which features a \mathcal{T}' family symmetry is discrete anomaly free and thus consistent with the assumed light particle content if

$$\sum_{i=\text{light}} Y_i \cdot \tilde{\ell}_i = 0 \bmod 24, \quad (3)$$

is satisfied for the discrete indices shown in Table 1, with $\xi = 0, 1$. We emphasize that the discrete anomaly condition of Eq. (3) is independent of the possible embeddings into $SU(2)$ or $SU(3)$.

Let us now apply Eq. (3) to some existing models of flavor. Demanding a grand unified structure in which all particles of the $SU(5)$ multiplets transform identically under \mathcal{T}' , the models of Refs. [5,6] are automatically discrete anomaly free because the sum of the hypercharges vanishes within each multiplet. In Refs. [7–10] the only particles contributing to the discrete anomaly of Eq. (3) are the quarks and leptons. Their assignments to irreps of \mathcal{T}' are given in Table 2, showing that the models of Refs. [7–9] are discrete anomaly free, while the one in Ref. [10] is anomalous and therefore incomplete.⁶

\mathcal{T}' models	Q	u^c	d^c	L	e^c	$\sum Y_i \tilde{\ell}_i$
Ref. [7]	$\mathbf{1}_0, \mathbf{2}_2$	$\mathbf{1}_0, \mathbf{2}_2$	$\mathbf{1}_0, \mathbf{2}_2$	$\mathbf{3}$	$\mathbf{1}_0, \mathbf{1}_1, \mathbf{1}_2$	0 mod 24
Ref. [8]	$\mathbf{1}_0, \mathbf{2}_0$	$\mathbf{1}_0, \mathbf{2}_1$	$\mathbf{1}_1, \mathbf{2}_2$	$\mathbf{3}$	$\mathbf{1}_0, \mathbf{1}_1, \mathbf{1}_2$	0 mod 24
Ref. [9]	$\mathbf{1}_2, \mathbf{2}_1$	$\mathbf{1}_1, \mathbf{2}_0$	$\mathbf{1}_1, \mathbf{2}_0$	$\mathbf{3}$	$\mathbf{1}_0, \mathbf{1}_1, \mathbf{1}_2$	0 mod 24
Ref. [10]	$\mathbf{1}_0, \mathbf{2}_0$	$\mathbf{1}_0, \mathbf{1}_1, \mathbf{1}_2$	$\mathbf{1}_1, \mathbf{2}_2$	$\mathbf{3}$	$\mathbf{1}_0, \mathbf{1}_1, \mathbf{1}_2$	12 mod 24

Table 2: The $\mathcal{T}' - \mathcal{T}' - U(1)_Y$ anomaly for various flavor models adopting the shown assignments of the quarks and leptons to irreps of \mathcal{T}' .

⁵See Footnote 5 of Ref. [3] for a discussion of the convention concerning the hypercharge normalization.

⁶Shortly after submitting this article to the archive, Ref. [10] was withdrawn by the authors.

Before turning to the binary dihedral groups \mathcal{Q}_{2n} , we should briefly remark on possible remnants of the Witten $SU(2)$ anomaly [23]. Mathematical consistency of the theory requires the number of left-handed fermionic $SU(2)$ doublets to be even if higher $SU(2)$ irreps are absent. Allowing for arbitrary $SU(2)$ irreps, this statement generalizes to the condition that the number of left-handed fermionic irreps with odd Dynkin index must be even, see e.g. Ref. [24]. These are the $SU(2)$ irreps of dimension $2+4m$ with $m \in \mathbb{N}$. As these $SU(2)$ irreps are the only ones that, under T' , decompose with an odd number of $\mathbf{2}_0$ irreps, the constraint from the Witten anomaly breaks down to the requirement of having an even number of $\mathbf{2}_0$ irreps in the complete theory, including the heavy degrees of freedom. From the Kronecker products we see that a heavy Majorana particle transforming as a $\mathbf{2}_0$ of T' is possible. Therefore the constraint from the Witten anomaly does not leave its footprints on the light particle content of a T' symmetric theory.

3 The Binary Dihedral Groups \mathcal{Q}_{2n}

The dihedral group \mathcal{D}_n is the symmetry group of the planar n -polygon. Its $2n$ elements can be expressed as rotations in three-dimensional space, indicating that $\mathcal{D}_n \subset SO(3)$. The dicyclic or binary dihedral group \mathcal{Q}_{2n} is the double cover of \mathcal{D}_n . It is defined by the presentation [22, 25]

$$\langle r, a, b \mid r^2 = 1, a^n = b^2 = r, aba = b \rangle . \quad (4)$$

Setting the auxiliary generator r to one, we recover the presentation of \mathcal{D}_n . The irreps of the binary dihedral group \mathcal{Q}_{2n} are

$$\begin{aligned} \mathbf{1}_{0,1} : \quad a &= 1, \quad b = \pm 1, \\ \mathbf{1}_{2,3} : \quad a &= -1, \quad b = \begin{cases} \pm 1 & (n = \text{even}) , \\ \pm i & (n = \text{odd}) , \end{cases} \\ \mathbf{2}_k : \quad a &= \begin{pmatrix} \eta^k & 0 \\ 0 & \eta^{-k} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}, \end{aligned}$$

where $\eta = e^{\pi i/n}$. The parameter k labels the two-dimensional representations and can formally take any integer value. However, a closer look at the generators reveals that

$$\mathbf{2}_{-k} = \mathbf{2}_k = \mathbf{2}_{k+2n}, \quad (5)$$

denote identical representations. Furthermore, $k = 0$ and $k = n$ lead to reducible representations:

$$\mathbf{2}_0 = \mathbf{1}_0 + \mathbf{1}_1, \quad \mathbf{2}_n = \mathbf{1}_2 + \mathbf{1}_3. \quad (6)$$

Therefore there are $n-1$ inequivalent irreducible two-dimensional representations, labeled by $k = 1, \dots, n-1$. The irreps of \mathcal{D}_n can be easily identified as

those $\mathbf{2}_k$ with even k , including $\mathbf{2}_0$ and (if n is even) $\mathbf{2}_n$. This shows that, for odd n , \mathcal{D}_n has only two one-dimensional irreps.

The Kronecker products for the irreps of \mathcal{Q}_{2n} are

$$\mathbf{1}_1 \otimes \mathbf{1}_1 = \mathbf{1}_0, \quad \mathbf{1}_1 \otimes \mathbf{1}_2 = \mathbf{1}_3, \quad \mathbf{1}_1 \otimes \mathbf{1}_3 = \mathbf{1}_2,$$

$$\mathbf{1}_2 \otimes \mathbf{1}_2 = \mathbf{1}_3 \otimes \mathbf{1}_3 = \begin{cases} \mathbf{1}_0 & (n = \text{even}) , \\ \mathbf{1}_1 & (n = \text{odd}) , \end{cases} \quad \mathbf{1}_2 \otimes \mathbf{1}_3 = \begin{cases} \mathbf{1}_1 & (n = \text{even}) , \\ \mathbf{1}_0 & (n = \text{odd}) , \end{cases}$$

$$\mathbf{1}_1 \otimes \mathbf{2}_k = \mathbf{2}_k, \quad \mathbf{1}_2 \otimes \mathbf{2}_k = \mathbf{1}_3 \otimes \mathbf{2}_k = \mathbf{2}_{n-k}, \quad \mathbf{2}_k \otimes \mathbf{2}_1 = \mathbf{2}_{k+1} + \mathbf{2}_{k-1},$$

where we make use of the identities in Eqs. (5) and (6). It is worth mentioning that the products involving $\mathbf{2}_k$ remain valid for $k = 0$ or n .

We now wish to embed \mathcal{Q}_{2n} into $SU(2)$. Since the determinant of the generator b for two-dimensional irreps depends on k , only $\mathbf{2}_k$ with an odd value for k can be identified with the $\mathbf{2}$ of $SU(2)$. They are the spinor-like irreps of \mathcal{Q}_{2n} , whereas the $\mathbf{2}_k$ with even k are vector-like. Furthermore, we want to embed the whole group \mathcal{Q}_{2n} and not just a subgroup of it. For that reason, the $\mathbf{2}$ (which generates all other $SU(2)$ irreps by successive multiplication) must correspond to $\mathbf{2}_\alpha$ with α and $2n$ being coprime, i.e. they have no common prime factor. For example, with $n = 10$ there are four possibilities, namely $\alpha = 1, 3, 7, 9$. Having identified the $\mathbf{2}$ of $SU(2)$ with $\mathbf{2}_\alpha$ of \mathcal{Q}_{2n} , the decomposition of the other irreps of $SU(2)$ is obtained from the Kronecker products. Using Eqs. (5) and (6), one finds the following embedding [26].⁷

Irreps ρ of $SU(2)$	Decomposition of ρ under \mathcal{Q}_{2n}	Dynkin index $\ell(\rho)$
2	$\mathbf{2}_\alpha$	1
3	$\mathbf{1}_1 + \mathbf{2}_{2\alpha}$	4
4	$\mathbf{2}_\alpha + \mathbf{2}_{3\alpha}$	10
5	$\mathbf{1}_0 + \mathbf{2}_{2\alpha} + \mathbf{2}_{4\alpha}$	20
6	$\mathbf{2}_\alpha + \mathbf{2}_{3\alpha} + \mathbf{2}_{5\alpha}$	35
7	$\mathbf{1}_1 + \mathbf{2}_{2\alpha} + \mathbf{2}_{4\alpha} + \mathbf{2}_{6\alpha}$	56

Again the Dynkin indices of $SU(2)$ are shown in order to help extract the discrete indices $\tilde{\ell}(\mathbf{r})$ of the irreps of \mathcal{Q}_{2n} . Setting $\tilde{\ell}(\mathbf{2}_0) = \tilde{\ell}(\mathbf{1}_1) = x$, it is an easy matter to prove that, in a compact notation,

$$\tilde{\ell}(\mathbf{2}_{\kappa\alpha}) = \kappa^2 + \frac{x}{2} \cdot (i^\kappa + i^{-\kappa}), \quad (7)$$

⁷The constraints arising from the Witten anomaly require that the number of spinor-like irreps, i.e. the $\mathbf{2}_{\kappa\alpha}$ with $\kappa = \text{odd}$, is even if all degrees of freedom are counted. However, the Kronecker products show that these spinor-like \mathcal{Q}_{2n} irreps can form Majorana mass term. Therefore, as in the case of \mathcal{T}' , no constraint on the light particle content is obtained from the Witten anomaly.

is consistent for the decomposition of all $SU(2)$ irreps. Notice that for odd κ the x -dependence drops out, while even κ entails the term xi^κ in addition to κ^2 . Of course, the discrete indices are only defined modulo N_ℓ . In order to determine its value we have to recall that $\mathbf{2}_{(\mathbf{n}+1)\alpha} = \mathbf{2}_{(\mathbf{n}-1)\alpha}$ so that their discrete indices must be identical, i.e.

$$\tilde{\ell}(\mathbf{2}_{(\mathbf{n}+1)\alpha}) - \tilde{\ell}(\mathbf{2}_{(\mathbf{n}-1)\alpha}) = 4nl + \frac{x}{2} \cdot (i^{n+l} + i^{-n-l} - i^{n-l} - i^{-n+l}) = 0 \bmod N_\ell ,$$

for all $l = 1, \dots, n$. This results in $N_\ell = 4n$. While the sum in the parentheses vanishes for even n , it is non-vanishing for odd n . In the latter case, x is therefore additionally constrained to be either 0 or $2n$, so that the discrete indices for the irreps of the binary dihedral group \mathcal{Q}_{2n} with odd n are given by

$$\tilde{\ell}(\mathbf{2}_{\kappa\alpha}) = \kappa^2 + \xi \cdot n (i^\kappa + i^{-\kappa}) \quad (n = \text{odd}) , \quad (8)$$

with $\xi = 0, 1$. For even n , Eq. (7) remains unchanged. Notice that the discrete indices for $\mathbf{2}_{\kappa\alpha}$, $\mathbf{2}_{-\kappa\alpha}$, and $\mathbf{2}_{(\kappa+2n)\alpha}$ are identical as required by the identities of Eq. (5). We emphasize that different embeddings of \mathcal{Q}_{2n} into $SU(2)$ are distinguished by α and have different discrete indices for a given irrep $\mathbf{2}_\mathbf{k}$. As for the indices of the one-dimensional irreps we remark that $\ell(\mathbf{1}_1) = \ell(\mathbf{2}_0)$ while $\tilde{\ell}(\mathbf{1}_2) + \tilde{\ell}(\mathbf{1}_3) = \tilde{\ell}(\mathbf{2}_\mathbf{n} = \mathbf{2}_{\mathbf{n}\alpha})$. This introduces a new parameter y into the definition of the discrete indices

$$\tilde{\ell}(\mathbf{1}_2) = y , \quad \tilde{\ell}(\mathbf{1}_3) = \tilde{\ell}(\mathbf{2}_{\mathbf{n}\alpha}) - y . \quad (9)$$

Alternatively, the discrete group \mathcal{Q}_{2n} could originate from $SU(3)$. In that case, the embedding would be defined by fixing how the $\mathbf{3}$ of $SU(3)$ breaks into irreps of \mathcal{Q}_{2n} . As we want to embed the complete group, the decomposition of the $\mathbf{3}$ must involve the irrep $\mathbf{2}_\alpha$ with α coprime to $2n$. The requirement of the generators a and b having determinant one then leads to the decomposition

$$\mathbf{3} \rightarrow \mathbf{1}_0 + \mathbf{2}_\alpha .$$

With the methods of Ref. [3] it is straightforward to prove that the resulting discrete indices are identical to Eqs. (7) and (9). Thus it is irrelevant whether the discrete symmetry \mathcal{Q}_{2n} originates in $SU(2)$ or $SU(3)$.

Finally, we need to discuss the particles which acquire mass when the continuous family symmetry is broken to \mathcal{Q}_{2n} and their effect on the $\mathcal{Q}_{2n} - \mathcal{Q}_{2n} - U(1)_Y$ anomaly. A look at the Kronecker products reveals that all but one of the bilinear mass terms are obtained from a *square* $\mathbf{r} \otimes \mathbf{r}$, so that they do not contribute to the discrete anomaly condition. The only exception is $\mathbf{1}_2 \otimes \mathbf{1}_3 = \mathbf{1}_0$ for odd values of n . In order for this not to change the discrete anomaly equation, we must choose y such that $\tilde{\ell}(\mathbf{1}_2) = \tilde{\ell}(\mathbf{1}_3)$. Then Eq. (9) gets replaced by

$$\tilde{\ell}(\mathbf{1}_2) = \tilde{\ell}(\mathbf{1}_3) = \frac{n^2}{2} + \zeta \cdot 2n \quad (n = \text{odd}) , \quad (10)$$

with $\zeta = 0, 1$, while for even n we still have Eq. (9) with arbitrary y . With the discrete indices defined in Eqs. (7)-(10), a model is discrete anomaly free if

$$\sum_{i=\text{light}} Y_i \cdot \tilde{\ell}_i = 0 \bmod 4n , \quad (11)$$

is satisfied for at least one embedding, i.e. one particular value of α . Let us assume that there exists such an embedding so that Eq. (11) can be written as

$$\sum_{j=0}^3 c'_j \tilde{\ell}(\mathbf{1}_j) + \sum_{\kappa=1}^{n-1} c_\kappa \tilde{\ell}(\mathbf{2}_\kappa \alpha) = 0 \bmod 4n, \quad (12)$$

where the integer coefficients c'_j and c_κ are obtained by summing the hypercharges of all particles living in the corresponding irrep of \mathcal{Q}_{2n} . Plugging in the explicit expressions for the discrete indices, we get

$$\left. \begin{aligned} c'_1 x + c'_2 y + c'_3 (n^2 + x i^n - y) + \sum_{\kappa=1}^{n-1} c_\kappa [\kappa^2 + x/2 (i^\kappa + i^{-\kappa})] \\ c'_1 \xi 2n + (c'_2 + c'_3) (n^2/2 + \zeta 2n) + \sum_{\kappa=1}^{n-1} c_\kappa [\kappa^2 + \xi n (i^\kappa + i^{-\kappa})] \end{aligned} \right\} = 0 \bmod 4n,$$

where the first/second line is valid for even/odd n . This equation must hold for all possible values of $x, y \in \mathbb{R}$ and $\xi, \zeta = 0, 1$. For even n it follows that $c'_2 = c'_3 = c_n$ as well as $c'_1 + c'_3 i^n + \sum_{\kappa=1}^{n-1} c_\kappa (i^\kappa + i^{-\kappa})/2 = 0$. For odd n the term $(c'_2 + c'_3) \cdot n^2/2$ has to be integer allowing us to define $2c_n = c'_2 + c'_3$ with $c_n \in \mathbb{Z}$. Thus we can simplify our equation to

$$\left. \begin{aligned} \sum_{\kappa=1}^n c_\kappa \kappa^2 \\ c'_1 \xi 2n + \sum_{\kappa=1}^n c_\kappa [\kappa^2 + \xi n (i^\kappa + i^{-\kappa})] \end{aligned} \right\} = 0 \bmod 4n.$$

Next we multiply everything with α^2 and use the fact that α is necessarily odd so that $\alpha^2 n = n + (\alpha - 1)(\alpha + 1)n = n \bmod 4n$. Therefore, after multiplication, we can remove the factor α^2 in those terms which are proportional to n . Furthermore, observing that $(i^\kappa + i^{-\kappa}) = (i^{\alpha\kappa} + i^{-\alpha\kappa})$, we finally obtain

$$\left. \begin{aligned} \sum_{\kappa=1}^n c_\kappa (\alpha\kappa)^2 \\ c'_1 \xi 2n + \sum_{\kappa=1}^n c_\kappa [(\alpha\kappa)^2 + \xi n (i^{\alpha\kappa} + i^{-\alpha\kappa})] \end{aligned} \right\} = 0 \bmod 4n.$$

This is identical to the discrete anomaly equation for the “standard” embedding which uses $\alpha = 1$. Hence, it is sufficient to evaluate Eq. (12) with $\alpha = 1$! If a model is shown to be anomalous for this embedding, it is automatically anomalous for all other embeddings. Else, the model is discrete anomaly free. We summarize the discrete indices for the standard embedding in Table 3.

To conclude our discussion of the binary dihedral groups, we calculate the discrete anomaly for some flavor models which rely on the groups \mathcal{Q}_{2n} . The models of Refs. [16, 17] are discrete anomaly free due to the $SU(5) \times \mathcal{Q}_{2n}$ grand unified structure. For the remaining examples, the assignments of the quarks and leptons to the irreps of \mathcal{Q}_{2n} are listed in Table 4. Other fermions that are introduced in these models (like for example the Higgs doublets in supersymmetric models) give no net contribution to the $\mathcal{Q}_{2n} - \mathcal{Q}_{2n} - U(1)_Y$ anomaly. Using the indices of Table 3, one finds that the models of Refs. [12–15, 18–20] are discrete anomaly free (the assignment in Ref. [15] is compatible with an $SU(5) \times \mathcal{Q}_{2n}$ structure, Ref. [18] features a Pati-Salam $\times \mathcal{Q}_6$ compatible assignment), while

\mathcal{Q}_{2n} irreps	$\tilde{\ell}(\mathbf{r}) \quad [N_\ell = 4n]$	
	$n = \text{even}$	$n = \text{odd}$
$\mathbf{1}_0$	0	0
$\mathbf{1}_1$	x	$\xi \cdot 2n$
$\mathbf{1}_2$	y	$n^2/2 + \zeta \cdot 2n$
$\mathbf{1}_3$	$n^2 + x i^n - y$	$n^2/2 + \zeta \cdot 2n$
$\mathbf{2}_k$	$k^2 + x/2 \cdot (i^k + i^{-k})$	$k^2 + \xi \cdot n (i^k + i^{-k})$

Table 3: The discrete indices $\tilde{\ell}$ for the irreps \mathbf{r} of \mathcal{Q}_{2n} using the standard embedding ($\alpha = 1$). The discrete anomaly condition has to be satisfied for all possible values of $x, y \in \mathbb{R}$ and $\xi, \zeta = 0, 1$.

the models proposed in Refs. [11, 21] need additional light particles to compensate the non-vanishing discrete anomaly. Whether this can be achieved without contradicting bounds on hypercharged light exotic particles needs an extensive study of all the possible cases and is therefore left for future investigations.

Group	Refs.	Q	u^c	d^c	L	e^c	$\sum Y_i \tilde{\ell}_i$
\mathcal{Q}_4	[11]	$\mathbf{1}_0, \mathbf{1}_0, \mathbf{1}_0$	$\mathbf{1}_0, \mathbf{1}_0, \mathbf{1}_0$	$\mathbf{1}_0, \mathbf{1}_0, \mathbf{1}_0$	$\mathbf{1}_0, \mathbf{2}_1$	$\mathbf{1}_0, \mathbf{1}_1, \mathbf{1}_3$	$2 - 6y \bmod 8$
\mathcal{Q}_4	[19, 20]	$\mathbf{1}_1, \mathbf{1}_2, \mathbf{1}_3$	$\mathbf{1}_1, \mathbf{1}_2, \mathbf{1}_3$	$\mathbf{1}_1, \mathbf{1}_2, \mathbf{1}_3$	$\mathbf{1}_0, \mathbf{2}_1$	$\mathbf{1}_0, \mathbf{2}_1$	$0 \bmod 8$
\mathcal{Q}_6	[12]	$\mathbf{1}_0, \mathbf{1}_0, \mathbf{1}_0$	$\mathbf{1}_0, \mathbf{1}_0, \mathbf{1}_0$	$\mathbf{1}_0, \mathbf{1}_0, \mathbf{1}_0$	$\mathbf{1}_1, \mathbf{2}_1$	$\mathbf{1}_1, \mathbf{1}_2, \mathbf{1}_3$	$0 \bmod 12$
\mathcal{Q}_6	[13, 14]	$\mathbf{1}_0, \mathbf{2}_1$	$\mathbf{1}_0, \mathbf{1}_0, \mathbf{1}_0$	$\mathbf{1}_1, \mathbf{2}_2$	$\mathbf{1}_0, \mathbf{2}_1$	$\mathbf{1}_1, \mathbf{2}_2$	$0 \bmod 12$
\mathcal{Q}_6	[14]	$\mathbf{1}_0, \mathbf{1}_0, \mathbf{1}_0$	$\mathbf{1}_0, \mathbf{2}_1$	$\mathbf{1}_1, \mathbf{2}_2$	$\mathbf{1}_0, \mathbf{1}_0, \mathbf{1}_0$	$\mathbf{1}_1, \mathbf{2}_2$	$0 \bmod 12$
\mathcal{Q}_6	[18]	$\mathbf{1}_1, \mathbf{2}_1$	$\mathbf{1}_3, \mathbf{2}_2$	$\mathbf{1}_3, \mathbf{2}_2$	$\mathbf{1}_1, \mathbf{2}_1$	$\mathbf{1}_3, \mathbf{2}_2$	$0 \bmod 12$
\mathcal{Q}_6	[21]	$\mathbf{1}_1, \mathbf{2}_1$	$\mathbf{1}_2, \mathbf{2}_2$	$\mathbf{1}_2, \mathbf{2}_2$	$\mathbf{1}_0, \mathbf{2}_2$	$\mathbf{1}_0, \mathbf{2}_2$	$3 \bmod 12$
\mathcal{Q}_{2n}	[15]	$\mathbf{1}_1, \mathbf{2}_2$	$\mathbf{1}_1, \mathbf{2}_2$	$\mathbf{1}_0, \mathbf{2}_1$	$\mathbf{1}_0, \mathbf{2}_1$	$\mathbf{1}_1, \mathbf{2}_2$	$0 \bmod 4n$

Table 4: The $\mathcal{Q}_{2n} - \mathcal{Q}_{2n} - U(1)_Y$ anomaly for various flavor models adopting the shown assignments of the quarks and leptons to irreps of \mathcal{Q}_{2n} .

4 Conclusion

In this letter we have derived the discrete indices for the irreps of the binary tetrahedral group \mathcal{T}' as well as the binary dihedral groups \mathcal{Q}_{2n} . Despite the ambiguities of embedding the finite symmetries into continuous $SU(2)$ and $SU(3)$, it is possible to define discrete indices that enter the discrete anomaly equations in a non-ambiguous way. Using the results shown in Tables 1 and 3, it is straightforward to check whether a given flavor model is consistent with a gauge origin of the applied discrete symmetry.

Having discussed the procedure for the groups \mathcal{T}' and \mathcal{Q}_{2n} , it should be clear how to obtain discrete indices for other subgroups of $SU(2)$ like the binary octahedral group and the binary icosahedral group. However, we are unaware of any example that imposes one of these groups as a family symmetry.

Note added: There has been some confusion as to whether the results of Ref. [27] and Ref. [3] are compatible. While Ref. [27] determines the discrete anomaly by calculating how the path integral changes under a transformation of the finite group \mathcal{G} , the approach pursued in Ref. [3] (and also in this article) relies on embedding \mathcal{G} into the continuous group $G_f = SU(3)$, $SO(3)$ or $SU(2)$. Due to the latter, an anomaly of the form $G_f - \text{SM} - \text{SM}$ is automatically absent if *complete* irreps of G_f are considered. On the other hand, the study in Ref. [27] is independent of a specific embedding and hence does not know of complete or incomplete multiplets of G_f . Therefore no inconsistencies exist between both results.

Acknowledgments

I am greatly indebted to Pierre Ramond for countless enlightening discussions and many helpful comments. This work is supported by the University of Florida through the Institute for Fundamental Theory.

References

- [1] C. D. Froggatt and H. B. Nielsen. *Nucl. Phys.*, B147:277, 1979.
- [2] P. F. Harrison, D. H. Perkins, and W. G. Scott. *Phys. Lett.*, B530:167, 2002, hep-ph/0202074.
- [3] C. Luhn and P. Ramond. *JHEP*, 07:085, 2008, arXiv:0805.1736.
- [4] L. M. Krauss and F. Wilczek. *Phys. Rev. Lett.*, 62:1221, 1989.
- [5] A. Aranda, C. D. Carone, and R. F. Lebed. *Phys. Rev.*, D62:016009, 2000, hep-ph/0002044.

- [6] M.-C. Chen and K. T. Mahanthappa. *Phys. Lett.*, B652:34, 2007, arXiv:0705.0714.
- [7] F. Feruglio, C. Hagedorn, Y. Lin, and L. Merlo. *Nucl. Phys.*, B775:120, 2007, hep-ph/0702194.
- [8] P. H. Frampton and T. W. Kephart. *JHEP*, 09:110, 2007, arXiv:0706.1186.
- [9] G.-J. Ding. *Phys. Rev.*, D78:036011, 2008, arXiv:0803.2278.
- [10] P. H. Frampton and S. Matsuzaki. 2007, arXiv:0712.1544.
- [11] D. Chang, W.-Y. Keung, and G. Senjanovic. *Phys. Rev.*, D42:1599, 1990.
- [12] D. Chang, W. Y. Keung, S. Lipovaca, and G. Senjanovic. *Phys. Rev. Lett.*, 67:953, 1991.
- [13] P. H. Frampton and T. W. Kephart. *Phys. Rev.*, D51:1, 1995, hep-ph/9409324.
- [14] P. H. Frampton and T. W. Kephart. *Int. J. Mod. Phys.*, A10:4689, 1995, hep-ph/9409330.
- [15] P. H. Frampton and O. C. W. Kong. *Phys. Rev. Lett.*, 75:781, 1995, hep-ph/9502395.
- [16] P. H. Frampton and O. C. W. Kong. *Phys. Rev.*, D53:2293, 1996, hep-ph/9511343.
- [17] P. H. Frampton and O. C. W. Kong. *Phys. Rev. Lett.*, 77:1699, 1996, hep-ph/9603372.
- [18] K. S. Babu and J. Kubo. *Phys. Rev.*, D71:056006, 2005, hep-ph/0411226.
- [19] M. Frigerio, S. Kaneko, E. Ma, and M. Tanimoto. *Phys. Rev.*, D71:011901, 2005, hep-ph/0409187.
- [20] M. Frigerio. 2005, hep-ph/0505144.
- [21] Y. Kajiyama, E. Itou, and J. Kubo. *Nucl. Phys.*, B743:74, 2006, hep-ph/0511268.
- [22] A. D. Thomas and G. V. Wood. *Group Tables*, Shiva Publishing Ltd., Kent 1980.
- [23] E. Witten. *Phys. Lett.*, B117:324, 1982.
- [24] C.-Q. Geng, R. E. Marshak, Z.-Y. Zhao, and S. Okubo. *Phys. Rev.*, D36:1953, 1987.
- [25] A. Blum, C. Hagedorn, and M. Lindner. *Phys. Rev.*, D77:076004, 2008, arXiv:0709.3450.

- [26] Paul H. Frampton and A. Rasin. *Phys. Lett.*, B478:424, 2000, hep-ph/9910522.
- [27] T. Araki et al. *Nucl. Phys.*, B805:124, 2008, arXiv:0805.0207.